BUMPING SEQUENCES AND MULTISPECIES JUGGLING

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Abstract. Building on previous work by four of us (ABCN), we consider further generalizations of Warrington's juggling Markov chains. We first introduce "multispecies" juggling, which consist in having balls of different weights: when a ball is thrown it can possibly bump into a lighter ball that is then sent to a higher position, where it can in turn bump an even lighter ball, etc. We both study the case where the number of balls of each species is conserved and the case where the juggler sends back a ball of the species of its choice. In this latter case, we actually discuss three models: add-drop, annihilation and overwriting. The first two are generalisations of models presented in (ABCN) while the third one is new and its Markov chain has the ultra fast convergence property. We finally consider the case of several jugglers exchanging balls. In all models, we give explicit product formulas for the stationary probability and closed form expressions for the normalization factor if known.

1. Introduction

Several Markov chains studied in nonequilibrium statistical physics are known to have, despite nontrivial dynamics, an explicit and sometimes remarkably simple stationary state. The most famous examples of these are one-dimensional models of hopping particles such as the asymmetric exclusion process [5], where the stationary state satisfies the so-called matrix product representation [4] and the zero-range process, where the stationary state is factorised [8]. The main reason for this simplicity is the underlying combinatorial structure of these processes. Of the two examples mentioned above, a variant of the former known as the totally asymmetric simple exclusion process (TASEP), solved first in [6], has a rich combinatorial structure even when the system is generalised to include several types of particles. The latter system is known as the *multispecies TASEP*, and its stationary state has an explicit solution which comes from queueing theory [9].

The multispecies TASEP has the further exceptional property that the stationary state can also be calculated if the hopping probabilities of particles depend on their location, known as the *inhomogeneous multispecies TASEP*. This was first done for the three-species case in [3] and the result

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for arbitrary species has been announced in [12]. While the stationary state of the general inhomogeneous multispecies TASEP has an explicit description in principle, the actual formulas for the stationary probabilities can be considerably complicated.

In this paper, we will first study the multispecies variants of the basic juggling process introduced in [13] then extended to their inhomogeneous versions in [7, 1]. In contrast to the TASEP, as we will show in Theorem 5, the stationary probabilities and the partition function have elegant and compact expressions. We then study the multispecies variants of two other juggling processes, which were also introduced in [13], where the number of balls of each type can vary. In all of these cases, we prove analogous results; see Theorems 9 and 11. We also introduce a new model where the number of balls of each type can vary that we call the overwriting model. This model has the nice property that it converges to its stationary distribution in deterministic finite time.

The rest of the paper is organized as follows. In Section 2, we discuss in some detail the first model, the so-called Multispecies Juggling Markov Chain (MSJMC): Section 2.1 provides its definition and the expression for its stationary distribution, and Section 2.2 is devoted to the enriched chain. Other models with a fluctuating number of balls of each type (but with a finite state space) are considered in Section 3: we introduce the multispecies extension of the add-drop and the annihilation models studied in [1] in the respective Sections 3.1 and 3.2. In Section 3.3, we present the overwriting model. Finally, in Section 4, we describe another possible extension of the juggling Markov chain of [13], that involves several jugglers.

Remark 1. Our proofs were mainly obtained by a classic combinatorial approach which consists of introducing an enriched chain whose stationary distribution is simpler, and which yields the original chain by a projection or "lumping" procedure, see e.g. [11, Section 2.3.1]. Let us summarize this strategy. Suppose we have a Markov chain on the state space S (which will be a finite set in all cases considered here), with transition matrix P (which is a matrix with rows and columns indexed by S, such that all rows sum up to 1), and for which we want to find the stationary distribution, namely the (usually unique) row vector π whose entries sum up to 1 and such that $\pi P = \pi$. The idea is to introduce another "enriched" Markov chain on a larger state space $\tilde{S}$ with transition matrix $\tilde{P}$, which has the two following properties:

- its stationary distribution $\tilde{\pi}$ is "easy" to find (for instance we may guess and then check its general form because its entries are integers with nice factorizations, or monomials in some parameters of the chain),
- it *projects* to the original Markov chain in the sense that there exists an equivalence relation $\sim$ over $\tilde{S}$ such that S can be identified with $\tilde{S}/\sim$ (i.e. the set of equivalence classes of $\sim$), and such that the *lumping condition*

$$\sum_{y' \sim y} \tilde{P}_{x,y'} = P_{[x],[y]} \tag{1}$$

is satisfied for all x, y in $\tilde{S}$, where $[x] \in S$ denotes the equivalence class of x.

Then, it is straightforward to check that the stationary distribution π of the original Markov chain is given by

$$\pi_{[x]} = \sum_{x' \sim x} \tilde{\pi}_{x'}. \tag{2}$$

In principle, there may be a large number of terms in the right-hand side of (2), making the resulting stationary distribution π nontrivial.

For all the juggling models, we prove that the Markov chain is aperiodic and irreducible. This implies that the stationary distribution is unique. For the enriched chains, we do not prove irreducibility (even though all the chains are conjectured to be irreducible). Computing a stationary distribution and lumping is enough to get the unique stationary distribution of the original chain.

Most of the results of this paper have been previously announced in the conference proceeding [2].

2. Multispecies juggling

2.1. **Definition and stationary distribution.**

The first model that we consider in this paper, and for which we give the most details, is a "multi-species" generalization of the so-called Multivariate Juggling Markov Chain (MJMC) [1]. Colloquially speaking, the juggler is now using balls of different weights, and when a heavy ball collides with a lighter one, the lighter ball is bumped to a higher position, where it can itself bump a lighter ball, and so on, until a ball arrives at the topmost position. Should the reader find this model unrealistic, she may instead think of a lazy referee "juggling" with a stack of papers of varying priorities to review: every day the referee takes the paper on the top of the stack but, after spending his time on other duties, decides to postpone it to a later date, possibly bumping a less important paper further down the stack, etc. Formally, our *Multispecies Juggling Markov Chain (MSJMC)* is defined as follows.

Let T be a fixed positive integer, and $n_1, \ldots, n_T$ be a sequence of positive integers. The state space $St_{n_1,\ldots,n_T}$ of the MSJMC is the set of words on the alphabet $\{1, \ldots, T\}$ containing, for all $i = 1, \ldots, T$, n_i occurrences of the letter i (the letter 1 represents the heaviest ball and T the lightest one). Of course those words have length $n = n_1 + \cdots + n_T$, and there are $\binom{n}{n_1,\ldots,n_T}$ different states.

To understand the transitions, it is perhaps best to start with an example, by considering the word 132132 (i.e. $T = 3$, $n_1 = n_2 = n_3 = 2$). The first letter 1 corresponds to the ball received by the juggler: it can be thrown either directly to the rightmost position, i.e. to the top (resulting in the word 321321), or in the place of any lighter ball. Say we throw it in place of the first 2. This 2 can in turn be thrown either to the rightmost position (resulting in the word 311322), or in the place of a lighter ball on its right: here it can only "bump" the second 3, which in turn has no choice but to go to the rightmost position, resulting in the word 311223. This latter transition is represented on Figure 1.

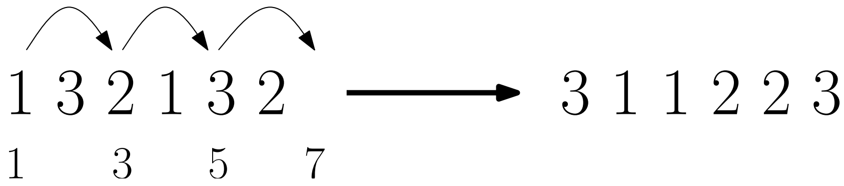

FIGURE 1. A possible transition from the state 132132, corresponding to the bumping sequence $(1, 3, 5, 7)$.

We now give a formal definition of the transitions. Let $w = w_1 \cdots w_n$ be a state in $St_{n_1,\ldots,n_T}$, and set, by convention, $w_{n+1} = \infty$. A *bumping sequence* for w is an increasing sequence of integers $(a(1), \ldots, a(k))$ with length at most $T + 1$ such that $a(1) = 1$, $a(k) = n + 1$ and, for all j between 1 and $k-1$, $w_{a(j)} < w_{a(j+1)}$ (that is to say, the ball at position $a(j)$ is heavier than that at position $a(j + 1)$). We denote by $\mathcal{B}_w$ the set of bumping sequences for w. For $a \in \mathcal{B}_w$, we define the state w^a resulting from w via the bumping sequence a by

$$w_i^a = \begin{cases} w_{a(\ell-1)} & \text{if } i = a(\ell) - 1 \text{ for some } \ell, \\ w_{i+1} & \text{otherwise,} \end{cases} \tag{3}$$

which is easily seen to belong to $St_{n_1,\ldots,n_T}$. Returning to the example in Figure 1 with $w = 132132$, the longest possible bumping sequence is $a = (1, 3, 5, 7)$ and indeed $w^a = 311223$.

We now turn to defining the transition probabilities, which means assigning a probability to each bumping sequence. As in the MJMC, these probabilities will depend on a sequence $z_1, z_2, \ldots$ of nonnegative real parameters, whose interpretation is now the following. Suppose that we have constructed the $i-1$ first positions $(a(1), \ldots, a(i-1))$ of a random bumping sequence, so that $a(i)$ has to be chosen in the set $\{\ell |\, a(i-1) < \ell \leqslant n+1, w_\ell > w_{a(i-1)}\}$: z_j is then proportional to the probability that we pick $a(i)$ as the j^{th} largest[1] element in that set. Upon normalizing, we find that the actual probability of picking a specific $a(i)$ can be written

$$Q_{w,a}(i) = \frac{z_{J_w(a(i), w_{a(i-1)})}}{y_{J_w(a(i-1), w_{a(i-1)})}}, \tag{4}$$

where we introduce the useful notation

$$y_i = z_1 + \cdots + z_i, \tag{5}$$

$$J_w(m, t) = 1 + \#\{\ell |\, m \leqslant \ell \leqslant n, w_\ell > t\}, \tag{6}$$

for $m \in \{1, \ldots, n\}$, $t \in \{1, \ldots, T\}$ and $i \in \{2, \ldots, k\}$. All in all, the global probability assigned to the bumping sequence a is $\prod_{i=2}^k Q_{w,a}(i)$. Noting that, for all states $w, w' \in St_{n_1,\ldots,n_T}$, there is at most one $a \in \mathcal{B}_w$ such that $w' = w^a$, we define the transition probability from w to w' as

$$P_{w,w'} = \begin{cases} \prod_{i=2}^k Q_{w,a}(i) & \text{if } w' = w^a \text{ for some } a \in \mathcal{B}_w, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

[1]If we replace “largest” by “smallest” here, then the MSJMC does not seem to have a simple stationary distribution anymore.

For instance, the transition of Figure 1 has probability $z_4/y_5 \times z_2/y_2 \times z_1/y_1$.

Remark 2. The MJMC [1] is recovered by taking $T = 2$, upon identifying 1's with balls ($\bullet$) and 2's with vacant positions ($\circ$).

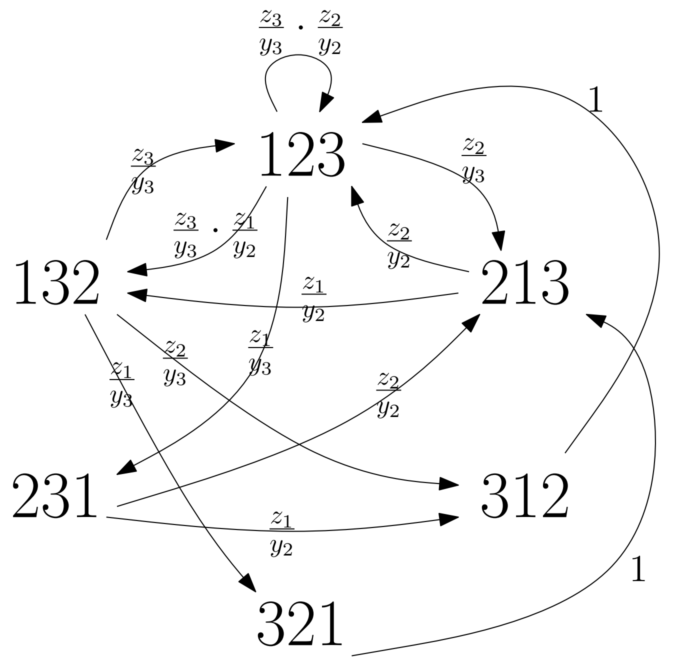

FIGURE 2. Transition graph of the MSJMC with $T = 3$ and $n_1 = n_2 = n_3 = 1$.

Example 3. Figure 2 illustrates the MSJMC on $St_{1,1,1}$, and the corresponding transition matrix in the basis $\{123, 132, 213, 231, 312, 321\}$ reads

$$\begin{pmatrix} \frac{z_3}{y_3} \cdot \frac{z_2}{y_2} & \frac{z_3}{y_3} \cdot \frac{z_1}{y_2} & \frac{z_2}{y_3} & \frac{z_1}{y_3} & 0 & 0 \\ \frac{z_3}{y_3} & 0 & 0 & 0 & \frac{z_2}{y_3} & \frac{z_1}{y_3} \\ \frac{z_2}{y_2} & \frac{z_1}{y_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{z_2}{y_2} & 0 & \frac{z_1}{y_2} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

Observe that $(y_1y_2y_3, y_1^2y_3, y_1y_2^2, y_1^2y_2, y_1^2y_2, y_1^3)$ is a left eigenvector with eigenvalue 1, and thus is proportional to the stationary distribution.

Remark 4. It is not difficult to check that generically (i.e. when all z_i are nonzero), the MSJMC is irreducible and aperiodic, and thus admits a unique stationary distribution π.

Indeed, we can reach any state π by the following procedure. Note that it suffices to reach a cyclic shift of π, since we can always throw balls to the rightmost position. Start by positioning the heaviest balls, that is the balls labelled 1 as is done in π. This can be done by the irreducibility of the unlabelled chain, studied in [1]. Assume now by induction that balls labelled $1, \ldots, i$ have been positioned as in π. The juggler will now throw all

balls labelled $1, \dots, i$ to the rightmost position and balls labelled $i+1$ to the positions given by π. Those positions will be occupied by lighter balls since all heavier balls allready are sorted according to π. Those lighter balls can bounce to anywhere, the rightmost position for instance. After T transitions all balls labelled $i+1$ have been sorted according to π. By induction we are done.
Furthermore, the state $1^{n_1}2^{n_2}\cdots T^{n_T}$ can be send to itself through the bumping sequence $(1, n_1+1, n_1+n_2+1, \dots, n+1)$, which proves the aperiodicity of the model.

Our main result for this section is an explicit expression for π.

Theorem 5. *The stationary probability of $w \in St_{n_1,\dots,n_T}$ is given by*

$$\pi(w) = \frac{1}{Z}\prod_{i=1}^{n} y_{E_w(i)} \tag{9}$$

with the notation

$$E_w(i) = 1 + \#\{j | i \leqslant j \leqslant n, w_j > w_i\} = J_w(i, w_i) \tag{10}$$

where the normalization factor Z reads

$$Z = \prod_{i=1}^{T} h_{n_i}(y_1, \dots, y_{n-n_1-\cdots-n_i+1}) \tag{11}$$

with h_ℓ the complete homogeneous symmetric polynomial of degree ℓ.

Returning again to the example $w = 132132$, we have $\pi(w) = y_1^3 y_2 y_3 y_5$. According to the general lumping strategy outlined in Remark 1, Theorem 5 is proved in Section 2.2 by introducing a suitable enriched Markov chain.

2.2. **The enriched Markov chain.** The first idea to define the enriched Markov chain comes from expanding the product on the right-hand side of (9) using the definition (5) of the y_j's, resulting in a sum of monomials in the z_j's which is naturally indexed by the set of sequences $v = v_1 \cdots v_n$ of positive integers such that $v_i \leqslant E_w(i)$ (with E as defined in (10)) for all $i \in \{1, \dots, n\}$. Let us call such v an *auxiliary word* for w. This suggests that we can define an enriched state as a pair (w, v) where $w \in St_{n_1,\dots,n_T}$ and v is an auxiliary word for w. We denote by $\mathcal{S}_{n_1,\dots,n_T}$ the set of enriched states.

The second idea, needed to define the transitions, is to use the auxiliary word to "record" some information about the past, in such a way that all transitions leading to a given enriched state have the same probability (this will be a key ingredient in the proof of Theorem 6 below). More precisely, given an enriched state (w, v), we consider as before a bumping sequence $a \in \mathcal{B}_w$, and we define the resulting enriched state $(w, v)^a = (w', v')$ by updating of course the basic state as before, i.e. we set $w' = w^a$ as in (3), while we update the auxiliary word as

$$v'_i = \begin{cases} E_{w'}(i) & \text{if } i = a(\ell) - 1 \text{ for some } \ell, \\ v_{i+1} & \text{otherwise.} \end{cases} \tag{12}$$

For instance, in our running example $w = 132132$ and $a = (1, 3, 5, 7)$, for $v = 412211$ we have $v' = 142211$. We may think of the auxiliary word as

“labels” carried by the balls, that are modified (maximized) for the bumped balls and preserved otherwise. The transition probability from (w,v) to (w',v') is as before

$$\widetilde{P}_{(w,v),(w',v')} = \begin{cases} \prod_{i=2}^{k} Q_{w,a}(i) & \text{if } (w',v') = (w,v)^a \text{ for some } a \in \mathcal{B}_w, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

It is clear that the enriched chain projects to the MSJMC. Indeed, we define an equivalence relation over $\mathcal{S}_{n_1,\ldots,n_T}$ by simply “forgetting” the auxiliary word, so that the equivalence classes may be identified with $St_{n_1,\ldots,n_T}$ (note that 1^n is a valid auxiliary word for any element of $St_{n_1,\ldots,n_T}$). The lumping condition (1) is trivially satisfied, since we have $\widetilde{P}_{(w,v),(w,v)^a} = P_{w,w^a}$ for all (w,v) in $\mathcal{S}_{n_1,\ldots,n_T}$ and a in $\mathcal{B}_w$, and $\widetilde{P}_{(w,v),(w^a,v')} = 0$ whenever $(w^a,v') \neq (w,v)^a$.

Theorem 6. *The stationary distribution of (w,v) in $\mathcal{S}_{n_1,\ldots,n_T}$ for the enriched chain is*

$$\tilde{\pi}(w,v) = \frac{1}{Z}\prod_{i=1}^{n} z_{v_i} \tag{14}$$

where Z is the normalization factor.

Proof. We have to check that, for all $(w',v') \in \mathcal{S}_{n_1,\ldots,n_T}$, we have

$$\sum_{(w,v)\in\mathcal{S}_{n_1,\ldots,n_T}} \widetilde{P}_{(w,v),(w',v')}\tilde{\pi}(w,v) = \tilde{\pi}(w',v'), \tag{15}$$

which is done by characterizing the possible predecessors of (w',v'). Let (w,v) be such that $(w',v') = (w,v)^a$ for some bumping sequence $a \in \mathcal{B}_w$. We will show in particular that a and w are uniquely determined from the data of (w',v') hence, as claimed above, all transitions to (w',v') have the same probability.

We start by explaining how to recover the bumping sequence $a = (a(1), \ldots, a(k))$ or, more precisely, its set of values $A = \{a(1),\ldots,a(k)\}$. Recall that 1 and $n+1$ belong to A by definition. We claim that $j \in \{2,\ldots,n\}$ belongs to A if and only if the following two conditions hold:

- (i) $v'_{j-1} = E_{w'}(j-1)$,
- (ii) $w'_{j-1} < w'_{j'-1}$ where j' is the smallest element of $A \cap \{j+1,\ldots,n+1\}$.

Indeed, these two conditions are clearly necessary: (i) by (12), and (ii) by (3) and the requirement that $w_j < w_{j'}$ when $j < j'$ are both in the bumping sequence. Conversely, assume that $j \notin A$, so that $w_j = w'_{j-1}$ and $v_j = v'_{j-1}$. By the definition of the MSJMC transitions, the subword $w'_j \cdots w'_n$ is a permutation of $w_{j+1}\cdots w_n w'_{j'-1}$. Hence, recalling (6), $E_{w'}(j-1) - E_w(j)$ is equal to 1 if (ii) holds and to 0 otherwise. If (i) holds, we have $E_{w'}(j-1) = v'_{j-1} = v_j \leqslant E_w(j)$, hence (ii) cannot hold. This completes the proof of our claim, which fully determines A (hence a) by reverse induction.

Once we have recovered a, it is clear that w is uniquely determined, while we have $v_j = v'_{j-1}$ for $j \notin A$. All predecessors of (w',v') are then obtained by picking, for each $j \in A\backslash\{n+1\}$, v_j an arbitrary integer between 1 and

$E_w(j)$. This shows that

$$\sum_{v:(w',v')=(w,v)^a} \tilde{\pi}(w,v) = \frac{1}{Z} \prod_{j \notin A} z_{v'_{j-1}} \prod_{j \in A \setminus \{n+1\}} y_{E_w(j)}. \tag{16}$$

The last observation we need is that

$$J_w(a(i), w_{a(i-1)}) = J_{w'}(a(i), w'_{a(i)-1}) = E_{w'}(a(i)-1) = v'_{a(i)-1} \tag{17}$$

for all $i \in \{2, \ldots, k\}$, since $w'_{a(i)} \cdots w'_n$ is a permutation of $w_{a(i)} \cdots w_n$ and since $w_{a(i-1)} = w'_{a(i)-1}$. By (4) and (7) we find that, for any predecessor (w,v) of (w',v'),

$$\widetilde{P}_{(w,v),(w',v')} = \frac{\prod_{j \in A \setminus \{1\}} z_{v'_{j-1}}}{\prod_{j \in A \setminus \{n+1\}} y_{E_w(j)}}. \tag{18}$$

Combined with (16), the desired stationarity condition (15) follows. □

Proof of Theorem 5. The expression (9) is immediately obtained by applying the general lumping expression (2) for the stationary state, Theorem 6 and the definition of enriched states. It remains to check the expression (11), which we do by induction on T. Let $\phi(w) = \prod_{i=1}^n y_{E_w(i)}$, so that Z is the sum of $\phi(w)$ over all $w \in St_{n_1,\ldots,n_T}$. The expression (11) holds for $T = 0$, as $Z = \phi(\epsilon) = 1$ where ϵ is the empty word. For $T \geqslant 1$, let w be a word in $St_{n_1,\ldots,n_T}$, and let $\hat{w} \in St_{n_2,\ldots,n_T}$ be the word obtained by removing all occurrences of 1 in w, and shifting all remaining letters down by 1. Denote by $i_1 > \cdots > i_{n_1}$ the positions of 1's in w, and let $j_\ell = n + 2 - i_\ell - \ell$, so that $1 \leqslant j_1 \leqslant \cdots \leqslant j_{n_1} \leqslant n - n_1 + 1$. The mapping $w \mapsto (\hat{w}, (j_1, \ldots, j_{n_1}))$ is bijective, and it is not difficult to see from the definition (10) of E that

$$\phi(w) = \phi(\hat{w}) \prod_{\ell=1}^{n_1} y_{j_\ell}. \tag{19}$$

Summing the product on the right-hand side over all sequences $(j_1, \ldots, j_{n_1})$ yields the complete homogeneous symmetric polynomial $h_{n_1}(y_1, \ldots, y_{n-n_1+1})$, and (11) follows by induction. □

3. Multispecies juggling with fluctuating types

Our goal is here to introduce multispecies generalizations of the add-drop and annihilation models developed in [13, 1]. Both models have the same state space and the same transition graph, but different transition probabilities. The state space is St_n^T, the set of words of length n on the alphabet $\mathcal{A} = \{1, \ldots, T\}$. The number of balls of each type is not fixed anymore and thus there are T^n possible states. The transitions are similar to the ones in the MSJMC, except that the type of the ball the juggler throws is independent of the type of the ball she just caught. This ball then initiates a bumping sequence as defined before. More precisely, starting with a state $w = w_1 \cdots w_n \in St_n^T$, we let $w^- = w_2 \cdots w_n$. Transitions involve replacing the first letter of w by an arbitrary $j \in \mathcal{A}$, resulting in the intermediate state jw^-, then applying a bumping sequence $a \in \mathcal{B}_{jw^-}$, resulting in the final state $(jw^-)^a$, where $(\cdot)^a$ is defined as in (3). Defining transitions probabilities requires specifying how we pick j and a. The multispecies add-drop

and annihilation models differ in the way that we pick the new ball of type j and the position $a(2)$ where it is inserted, while the subsequent elements $a(3), \dots, a(k)$ of the bumping sequence are then chosen in the same way as for the MSJMC. Figure 3 shows all allowed transitions for St_2^3.

Remark 7. Both chains are irreducible, since a state $w_1 \cdots w_n$ can be reached from any state in n steps by just putting a w_i in the rightmost position at the i-th step. The models are also aperiodic since the state 1^n can be send to itself by putting a 1 at the rightmost position.This remark also applies to the overwriting model, described later in Section 3.3.

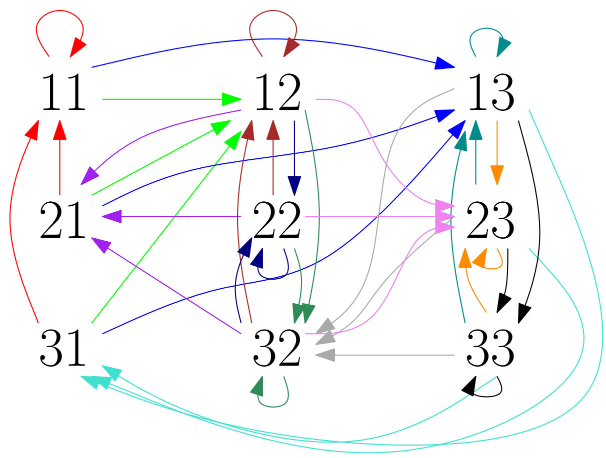

FIGURE 3. The basic transition graph on St_2^3. Note that the first letter has no effect on which states can be reached.

3.1. **Add-drop model.** In the add-drop model, choosing a ball of type j and sending it to the ℓ-th available position from the right is done with probability proportional to $c_j z_\ell$ where, in addition to the previous parameters $z_1, z_2, \dots$, we introduce new nonnegative real parameters $c_1, \dots, c_T$ that can be interpreted as "activities" for each type of ball. Because lighter balls can be inserted in fewer possible positions, the actual probability of choosing j and ℓ has to be normalized, and reads $c_j z_\ell / (\sum_{t=1}^T c_t y_{J_w(2,t)})$ where w is the initial state and where we use the same notations (5) and (6) as before (note that $J_w(m,t) = J_{jw^-}(m,t)$ for all $m > 1$). As the position where the new ball is inserted is $a(2) > 1$, saying that it is the ℓ-th available position from the right means that $\ell = J_w(a(2), j)$. As described above, subsequent elements $a(3), \dots, a(k)$ of the bumping sequence a are chosen in the same way as for the MSJMC, so that the global probability of picking a new ball of type j and a bumping sequence $a \in \mathcal{B}_{jw^-}$ is

$$p_w(j,a) = \frac{c_j z_{J_w(a(2),j)}}{\sum_{t=1}^T c_t y_{J_w(2,t)}} \prod_{i=3}^k Q_{w,a}(i) \tag{20}$$

where we recall the notation (4). The *multispecies add-drop juggling Markov chain* is then the Markov chain on the state space St_n^T for which the transition

probability from w to w' reads

$$P_{w,w'} = \begin{cases} p_w(j,a) & \text{if } w' = (jw^-)^a \text{ for some } j \in \mathcal{A} \text{ and } a \in \mathcal{B}_{jw^-}, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Note that we recover the add-drop juggling model [1] when we set $T = 2$.

Example 8. The transition matrix of the multispecies add-drop Markov chain on the state space St_2^3 in the ordered basis $\{11, 21, 31, 12, 22, 32, 13, 23, 33\}$ reads

$$\begin{pmatrix}
\frac{c_1z_1}{\lambda_1} & 0 & 0 & \frac{c_2z_1}{\lambda_1} & 0 & 0 & \frac{c_3z_1}{\lambda_1} & 0 & 0 \\
\frac{c_1z_1}{\lambda_1} & 0 & 0 & \frac{c_2z_1}{\lambda_1} & 0 & 0 & \frac{c_3z_1}{\lambda_1} & 0 & 0 \\
\frac{c_1z_1}{\lambda_1} & 0 & 0 & \frac{c_2z_1}{\lambda_1} & 0 & 0 & \frac{c_3z_1}{\lambda_1} & 0 & 0 \\
0 & \frac{c_1z_1}{\lambda_2} & 0 & \frac{c_1z_2}{\lambda_2} & \frac{c_2z_1}{\lambda_2} & 0 & 0 & \frac{c_3z_1}{\lambda_2} & 0 \\
0 & \frac{c_1z_1}{\lambda_2} & 0 & \frac{c_1z_2}{\lambda_2} & \frac{c_2z_1}{\lambda_2} & 0 & 0 & \frac{c_3z_1}{\lambda_2} & 0 \\
0 & \frac{c_1z_1}{\lambda_2} & 0 & \frac{c_1z_2}{\lambda_2} & \frac{c_2z_1}{\lambda_2} & 0 & 0 & \frac{c_3z_1}{\lambda_2} & 0 \\
0 & 0 & \frac{c_1z_1}{\lambda_3} & 0 & 0 & \frac{c_2z_1}{\lambda_3} & \frac{c_1z_2}{\lambda_3} & \frac{c_2z_2}{\lambda_3} & \frac{c_3z_1}{\lambda_3} \\
0 & 0 & \frac{c_1z_1}{\lambda_3} & 0 & 0 & \frac{c_2z_1}{\lambda_3} & \frac{c_1z_2}{\lambda_3} & \frac{c_2z_2}{\lambda_3} & \frac{c_3z_1}{\lambda_3} \\
0 & 0 & \frac{c_1z_1}{\lambda_3} & 0 & 0 & \frac{c_2z_1}{\lambda_3} & \frac{c_1z_2}{\lambda_3} & \frac{c_2z_2}{\lambda_3} & \frac{c_3z_1}{\lambda_3}
\end{pmatrix} \tag{22}$$

with the notation $\lambda_1 = (c_1+c_2+c_3)y_1$, $\lambda_2 = c_1y_2+(c_2+c_3)y_1$ and $\lambda_3 = (c_1+c_2)y_2 + c_3y_1$. One can check that $(c_1^2y_1^2, c_1c_2y_1^2, c_1c_3y_1^2, c_1c_2y_1y_2, c_2^2y_1^2, c_2c_3y_1^2, c_1c_3y_1y_2, c_2c_3y_1y_2, c_3^2y_1^2)$ is a left eigenvector for the eigenvalue 1.

Theorem 9. *The stationary probability of $w = w_1 \cdots w_n \in St_n^T$ for the add-drop model is given by*

$$\pi(w) = \frac{1}{Z} \prod_{i=1}^{n} c_{w_i} y_{E_w(i)}. \tag{23}$$

where the normalization factor Z reads

$$Z = \sum_{n_1+\cdots+n_T=n} \left(c_1^{n_1} \cdots c_T^{n_T} \prod_{i=1}^{T} h_{n_i}(y_1, \ldots, y_{n-n_1-\cdots-n_i+1}) \right) \tag{24}$$

with h_ℓ the complete homogeneous symmetric polynomial of degree ℓ.

Proof. We will again follow the strategy described in Remark 1. We consider the enriched Markov chain whose state space is the set $\mathcal{S}_n^T$ of pairs of words (w, v) with $w \in St_n^T$ and $v = v_1 \cdots v_n$ such that $v_i \leqslant E_w(i)$ for all $i \in \{1, \ldots, n\}$. Given $(w, v) \in \mathcal{S}_n^T$, $j \in \mathcal{A}$ and $a \in \mathcal{B}_{jw^-}$, we define the resulting enriched state $(w, v)_j^a = (w', v')$ by setting $w' = (jw^-)^a$, and

$$v'_i = \begin{cases} E_{w'}(i) & \text{if } i = a(l) - 1 \text{ for some } l, \\ v_{i+1} & \text{otherwise.} \end{cases} \tag{25}$$

and the transition probabilities are of course given by :

(26)

$$\tilde{P}_{(w,v),(w',v')} = \begin{cases} p_w(j,a) & \text{if } (w',v') = (w,v)_j^a \text{ for some } j \in \mathcal{A} \text{ and } a \in \mathcal{B}_{jw^-}, \\ 0 & \text{otherwise.} \end{cases}$$

with p as defined in (20).

We will now show that the stationary probability of $(w, v) \in \mathcal{S}_n^T$ for the enriched add-drop model is given by

$$\tilde{\pi}(w, v) = \frac{1}{Z} \prod_{i=1}^{n} c_{w_i} z_{v_i}, \tag{27}$$

which will give us equation (23) by lumping.

We have to check that, for all $(w', v') \in \mathcal{S}_n^T$, we have

$$\sum_{(w,v) \in \mathcal{S}_n^T} \widetilde{P}_{(w,v),(w',v')} \tilde{\pi}(w, v) = \tilde{\pi}(w', v'). \tag{28}$$

Let (w', v') be a state in $\mathcal{S}_n^T$. For a given (w', v') we can deduce most of a possible predecessor (w, v). As in the proof of Theorem 6 we can first deduce the the bumping sequence a, then the type j of the added ball. This means that $w_2, \dots, w_n$ and $v_i, i \notin A$ are uniquely determined. Recall that for $a = (a(1), \dots, a(k))$ we defined $A = \{a(1), \dots, a(k)\}$. Let $W = jw_2 \cdots w_n$. We have :

$$\sum_{(w,v) : (w',v') = (w,v)_j^a} \tilde{\pi}(w, v) = \frac{1}{Z} \sum_{i=1}^{T} c_i y_{J_W(2,i)} \prod_{\ell \notin A} z_{v'_{\ell-1}} \prod_{\ell \in A \setminus \{1, n+1\}} y_{E_W(\ell)}. \tag{29}$$

Furthermore, by observing that $J_w(a(i), w_{a(i-1)}) = v'_{\ell-1}$ (as in equation (17)) we have, for all (w, v) such that $(w, v)_j^a = (w', v')$,

$$\tilde{P}_{(w,v),(w',v')} = \frac{c_j z_{J_W(a(2),j)}}{\sum_{t=1}^{T} c_t y_{J_W(2,t)}} \frac{\prod_{\ell \in A \setminus \{1, a(2)\}} z_{v'_{\ell-1}}}{\prod_{\ell \in A \setminus \{1, n+1\}} y_{E_W(\ell)}} \tag{30}$$

Combined with (29), the desired stationarity condition (28) follows. □

3.2. **Annihilation model.** In this section, we assume that hopping parameters are probabilities, $z_1 + \cdots + z_{n+1} = 1$. In this model, we consider that the juggler first tries to send a ball of type 1. She chooses $\ell \in \{1, \dots, n+1\}$ with probability z_ℓ, and tries to send the ball at the ℓ-th available position (counted from the right as before). If ℓ is a valid position (that is, it is not larger than the number of available positions for the 1), there is a bumping sequence whose subsequent elements are drawn in the same way as for the MSJMC. Otherwise, she tries instead to send a 2 according to the same procedure, etc. In the end, if she did not manage to send any ball of type in $\{1, \dots, T-1\}$, she just sends a T to the righmost position. Note that failing to send a ball of type t for an initial state w is done with probability

$$1 - y_{J_w(2,t)}. \tag{31}$$

Globally, the probability of picking a new ball of type j and a bumping sequence $a \in \mathcal{B}_{jw^-}$ reads

$$q_w(j, a) = \begin{cases} z_{J_w(a(2),j)} \prod_{t=1}^{j-1} \left(1 - y_{J_w(2,t)}\right) \prod_{i=3}^{k} Q_{w,a}(i) & \text{if } j < T, \\ \prod_{t=1}^{T-1} \left(1 - y_{J_w(2,t)}\right) & \text{if } j = T, \end{cases} \tag{32}$$

note that $j = T$ implies $a = (1, n+1)$. The transition probabilities of the *multispecies annihilation juggling Markov chain* are obtained by replacing $p_w(j,a)$ with $q_w(j,a)$ in (21). Note that we recover the annihilation juggling model [1] when we set $T = 2$.

Example 10. The transition matrix of the multispecies annihilation Markov chain on the set space St_2^3 in the ordered basis $\{11, 21, 31, 12, 22, 32, 13, 23, 33\}$ reads

$$\begin{pmatrix} z_1 & 0 & 0 & z_1(z_2+z_3) & 0 & 0 & (z_2+z_3)^2 & 0 & 0 \\ z_1 & 0 & 0 & z_1(z_2+z_3) & 0 & 0 & (z_2+z_3)^2 & 0 & 0 \\ z_1 & 0 & 0 & z_1(z_2+z_3) & 0 & 0 & (z_2+z_3)^2 & 0 & 0 \\ 0 & z_1 & 0 & z_2 & z_1z_3 & 0 & 0 & (z_2+z_3)z_3 & 0 \\ 0 & z_1 & 0 & z_2 & z_1z_3 & 0 & 0 & (z_2+z_3)z_3 & 0 \\ 0 & z_1 & 0 & z_2 & z_1z_3 & 0 & 0 & (z_2+z_3)z_3 & 0 \\ 0 & 0 & z_1 & 0 & 0 & z_1z_3 & z_2 & z_2z_3 & z_3^2 \\ 0 & 0 & z_1 & 0 & 0 & z_1z_3 & z_2 & z_2z_3 & z_3^2 \\ 0 & 0 & z_1 & 0 & 0 & z_1z_3 & z_2 & z_2z_3 & z_3^2 \end{pmatrix} \tag{33}$$

Note that $(z_1^2, z_1^2(z_2+z_3), z_1(z_2+z_3)^2, z_1(z_1+z_2)(z_2+z_3), z_1^2z_3(z_2+z_3), z_1z_3(z_2+z_3)^2, (z_1+z_2)(z_2+z_3)^2, z_3(z_1+z_2)(z_2+z_3)^2, z_3^2(z_2+z_3)^2)$ is a left eigenvector for the eigenvalue 1.

Theorem 11. *The stationary probability of $w = w_1 \cdots w_n \in St_n^T$ for the annihilation model is given by*

$$\pi(w) = \left(\prod_{i=1, w_i<T}^{n} y_{E_w(i)}\right)\left(\prod_{\ell=2}^{T}\prod_{p=1}^{\#\{m|w_m \geqslant \ell\}} (1-y_p)\right). \tag{34}$$

Moreover, here no normalization factor is needed as

$$\sum_{w\in St_n^T} \pi(w) = (z_1 + \cdots + z_{n+1})^{n(T-1)} = 1. \tag{35}$$

The stationary probabilities of enriched states are no longer monomials in the z_i's, which suggest that a further enrichment is possible as already observed for the case $T = 2$ in Section 4.2 of [1].

Proof. The theorem is proved by enriching the chain as before. We again use the state space $\mathcal{S}_n^T$ for the enriched multispecies annihilation Markov chain, just as for the enriched multispecies add-drop Markov chain in Section 3.1. The transitions are defined in the same way too, only the transition probabilities change :

$$\tilde{P}_{(w,v),(w',v')} = \begin{cases} q_w(j,a) & \text{if } (w',v') = (w,v)_j^a \text{ for some } j \in \mathcal{A} \text{ and } a \in \mathcal{B}_{jw^-}, \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

with q as defined in (32). We will now show that the stationary probability of $(w,v) \in \mathcal{S}_n^T$, $w = w_1 \cdots w_n$, $v = v_1 \cdots v_n$ is given by

$$\tilde{\pi}(w,v) = \prod_{i=1,w_i<T}^{n} z_{v_i} \prod_{\ell=2}^{T}\prod_{p=1}^{\#\{m|w_m\geqslant\ell\}} (1-y_p). \tag{37}$$

Once we prove this, we will obtain a proof of (34) by lumping. To do so, we have to check that $\tilde{\pi}$ satisfies

$$\sum_{(w,v)\in\mathcal{S}_n^T} \tilde{P}_{(w,v),(w',v')}\tilde{\pi}(w,v) = \tilde{\pi}(w',v'). \tag{38}$$

Let (w',v') be a state in $\mathcal{S}_n^T$. We do not have a lot of choice in choosing a predecessor (w,v) of (w',v'); the bumping sequence a, the type j of the added ball, $w_2,\dots,w_n$ and $v_i, i\notin A$, where as before A is the set of values in a, are uniquely determined (this works exactly as for the proof of Theorem 6). Let $W = jw_2\cdots w_n$. We define :

$$\begin{aligned}
C &= \prod_{i=1,w_i'<T}^{n} z_{v_i'} \\
C' &= \begin{cases} z_{J_W(a(2),j)} \dfrac{\prod_{i\in A\setminus\{1,a(2)\}} z_{v'_{i-1}}}{\prod_{i\in A\setminus\{1,n+1\}} y_{E_W(i)}} & \text{if } j<T, \\ 1 & \text{otherwise.} \end{cases} \\
C'' &= \prod_{\substack{i\in A\setminus\{1,n+1\}\\ w_i<T}} y_{E_W(i)} \prod_{\substack{i\notin A\\ w_i<T}} z_{v_i} \\
D &= \prod_{\ell=2}^{T} \prod_{p=1}^{\#\{m|w_m'\geqslant\ell\}} (1-y_p) \\
D' &= \prod_{i=1}^{j-1}(1-y_{J_W(2,i)}) \\
D'' &= \prod_{\ell=2}^{T} \prod_{p=1}^{\#\{m\geqslant 2|w_m\geqslant\ell\}} (1-y_p)
\end{aligned}$$

$$K = \sum_{w_1=1}^{T} y_{E_{w_1w_2\cdots w_n}(1)} \prod_{\ell=2}^{w_1}(1-y_{\#\{m\geqslant 2|w_m\geqslant\ell\}+1})$$

First, we note that $C'C''=C$. Recall that $j=T$ implies $A=\{1,n+1\}$. Secondly, we have $D'D''=D$, since $J_W(2,i) = \#\{m\geqslant 2|w_m>i\}+1$. For all (w,v) such that $\tilde{P}_{(w,v),(w',v')}\neq 0$, we have

$$\tilde{P}_{(w,v),(w',v')} = C'D', \tag{39}$$

where the C' follows from (17) and (4) and D' follows from (31). Note that the transfer probability does not depend on the choice of (w,v). Now,

$$\sum_{(w,v):(w',v')=(w,v)_j^a} \tilde{\pi}(w,v) = C''D''K, \tag{40}$$

where the $i=1$ case has been removed from C'' and collected in K. Furthermore, $K=1$, since it is telescoping sum using that $E_{w_1w_2\cdots w_n}(1) = \#\{m\geqslant 2|w_m>w_1\}+1$, (start with adding the cases $w_1=T$ and $w_1=T-1$ together). The desired condition (38) follows.

□

3.3. **Overwriting model.** The aim is here to describe a multispecies generalisation of the annihilation model studied in [1] in which the *ultrafast convergence* property holds, namely we want the stationary distribution to be reached in a finite number of steps, independent of the starting distribution. Let P be the transition matrix of a Markov chain (which is assumed to be irreducible and aperiodic). Saying that this Markov chain has the ultrafast convergence property is equivalent to saying that there exists an integer m such that P^m is the matrix whose rows are copies of the left eigenvector of P for the eigenvalue 1, or to saying that P has only one nonzero eigenvalue (which is 1).

3.3.1. *Model description.* The set space St_n^T is, as described in Section 3, is the set of words in $\{1, \ldots, T\}^n$, and the transition probabilities are dictated by the indeterminates $z_1, \ldots, z_{n+1}$ satisfying $z_1 + z_2 + \cdots + z_{n+1} = 1$ as in Section 3.2.

In the *overwriting model*, each integer gets a chance to overwrite an integer larger than it. More formally, the transitions are described by a the following process. Initially, the 1-st letter is erased, and everything is moved to the left by 1 step. Now the juggler first tries to send a ball of type 1; she chooses $i \in \{1, n+1\}$ with probability z_i, and aims for the i^{th} available position, meaning those positions which contain an integer greater than 1 (counting from right as before). If i is greater than the number of available positions for a 1 to land, the juggler simply fails to send a 1, otherwise the 1 lands to some position, destroying the ball previously occupying it if it isn't the righmost position. She then tries to send a 2, which can only land in a higher position than the 1 did (any available position if the 1 didn't land), or fail to land. She then tries to send a 3, and so on. If, after trying to send a ball of each type in $\{1, \ldots, T-1\}$, no ball landed to the rightmost position, she puts a T in that position. Otherwise, the new state is reached as soon as a ball is sent to the rightmost position.

For $w \in St_n^T$, we define an *overwriting sequence* $B = ((b_1, t_1), \ldots, (b_k, t_k))$ for w as follows : $1 < b_1 < b_2 < \ldots < b_k = n+1$, $1 \leqslant t_1 < t_2 < \ldots < t_k \leqslant T$, and for all $i = 1, \ldots, k$, $t_i < w_{b_i}$ with, by convention, $w_{n+1} = +\infty$. We denote by $\mathcal{B}_w$ the set of overwriting sequences for w.

For $w \in St_n^T$ and $B \in \mathcal{B}_w$, the state in St_n^T obtained by applying the overwriting sequence B to the word w, denoted w^B is defined as

$$w_i^B = \begin{cases} t_j & \text{if } i = b_j - 1 \text{ for some } j \\ w_{i+1} & \text{otherwise.} \end{cases} \tag{41}$$

Note that the probability for the juggler to fail to send a ball of type ℓ during the j-th part of the overwriting is given by

$$1 - y_{J_w(b_{j-1}+1,\ell)} \tag{42}$$

with the convention that $b_0 = 1$ and where J is as in (6) Note that this returns 1 if a ball has already been sent to the rightmost position. The probability for the juggler to succeed in sending a ball of type t_j to position $b_j - 1$ during the j-th part of the overwriting is given by

$$z_{J_w(b_j,t_j)}. \tag{43}$$

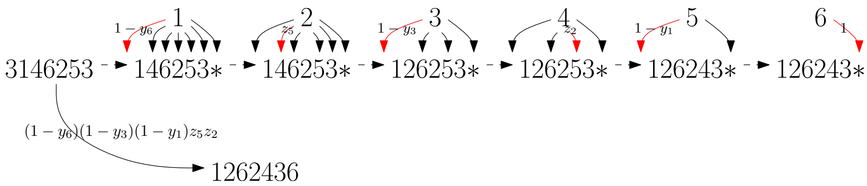

FIGURE 4. A transition in the overwriting multispecies Markov chain on St_7^6

Example 12. In Figure 4, we describe the transition from the state 3146253 to the state 1262436 in the state space St_7^6. The corresponding overwriting sequence is given by $((3,2),(6,4),(8,6))$ and the probability of this transition is $(1-y_6)z_5(1-y_3)z_2(1-y_1)$.

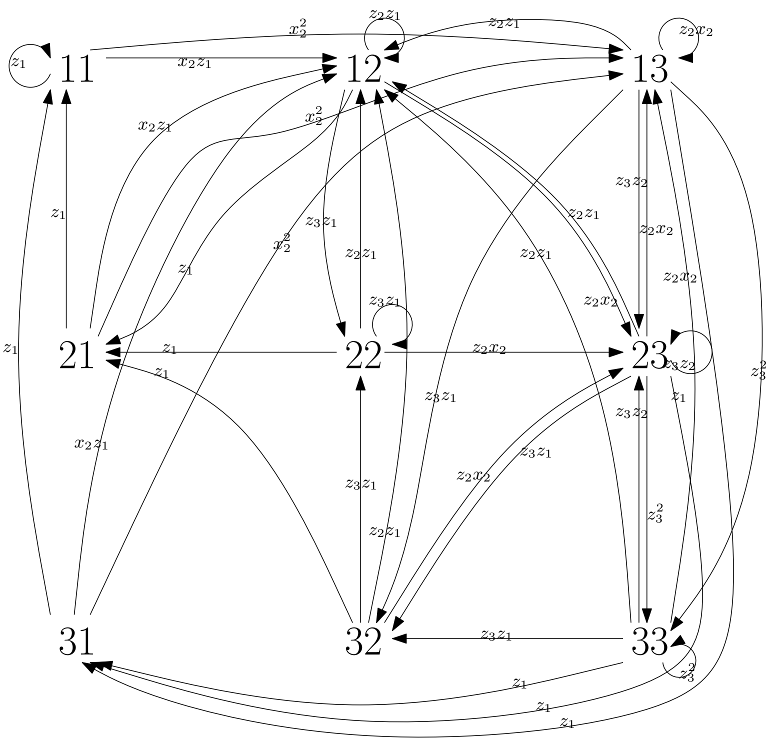

FIGURE 5. The overwriting multispecies Markov chain on St_2^3. Here $x_i = 1 - y_{i-1}$.

Given $z_1, \dots, z_{n+1}$ nonnegative real numbers summing to 1, we are now able to define the transition probabilities for the annihilation multispecies Markov chain : for $w, w' \in St_n^T$ if there exists $B \in \mathcal{B}_w$ such that $w' = w^B$, the transition probability from w to w' reads :

$$\mathcal{P}_{w,w'} = \prod_{j=1}^{k} \Big(\prod_{\ell=t_{j-1}+1}^{t_j-1} (1 - y_{J_w(b_{j-1}+1,\ell)}) \Big) \prod_{j|t_j \neq T} z_{J_w(b_j,t_j)}, \tag{44}$$

and it reads 0 otherwise.

Example 13. The transitions of the overwriting Markov chain on St_2^3 are represented in Figure 5.

The stationary distribution of the overwriting chain does not have a simple formula, unlike the add-drop and annihilation multispecies variants. However, we do obtain an indirect formula in terms of the enriched chain, which we state as Corollary 17 in the next section. It turns out that the occupation probability for the last site and the joint occupation distributions at the last two sites have particularly simple expressions, which is what we state next.

Theorem 14. *The stationary probability of having a j at the last site is given by*

$$\mathbb{P}(w_n = j) = \begin{cases} z_1(1-z_1)^{j-1} & \text{if } j < T, \\ (1-z_1)^{T-1} & \text{if } j = T. \end{cases}$$

The joint probability of having an i at the $n-1$-th site and a j at the n-th site is given by

$$\mathbb{P}(w_{n-1} = i,\, w_n = j) = (1-z_1)^{\max(i,j)-1}(1-y_2)^{\min(i,j)-1} \times \begin{cases} z_1 y_2 & \text{if } i < j < T, \\ z_1^2 & \text{if } j \leqslant i < T, \\ y_2 & \text{if } i < j = T, \\ z_1 & \text{if } j < i = T, \\ 1 & \text{if } i = j = T. \end{cases}$$

The proof is given in the following section, by constructing an enriched chain, and analysing the transitions therein.

3.3.2. *Staircase tableaux enrichment.* It is now natural to look at staircase tableaux, as was done for the original juggling model in [7]. The state space for the enriched version of the overwriting chain on St_n^T is the set of Young tableaux of shape $(n, n-1, \dots, 2, 1)$, with the following conditions on the entries in cells.

(1) Entries belong to the set $\{1, \dots, T-1\}$.
(2) Entries appear in increasing order from left to right, and from bottom to top.
(3) Empty cells are allowed.

This set of tableaux is denoted by $\mathcal{T}_n^T$. For $V \in \mathcal{T}_n^T$, we will denote by $V_{*,k}$ the k-th column, from the left, of V (which has $n+1-k$ cells).

The transitions of the enriched Markov chain are as follows. At each step, the entries in the bottom row are deleted, all remaining entries are moved one step down and one step right, and we add entries to the leftmost column so that the tableaux conditions above still hold. More precisely, we proceed in the following way. We first try to add a 1 by choosing a number k_1 in $\{1, \dots, n+1\}$ and placing a 1 in the k_1-th free position (from top to bottom; a position is "free" if and only if there is no 1 in the same row). If k_1 is greater than the number of free positions, no 1 is added. We then similarly try to add a 2 by choosing k_2 in $\{1, \dots, n+1\}$ and placing a 2 in the k_2-th free position, a position being free if there is no 1 or 2 in the same row, and having no 1 above it. We continue this way until all numbers between 1 and $T-1$ have been tried.

For $V \in \mathcal{T}_n^T$, for $i \in \{1, \ldots, T\}$ and $k \in \{1, \ldots, n\}$, we introduce the useful notation :

$$(45)\quad \mathcal{C}_V(i,k) = \begin{cases} z_{1+\#\{\text{cells above entry } i \text{ in } V_{*,k}, \text{ with no entry } j \leqslant i \text{ to the right}\}}, & \text{if } i \text{ is in } V_{*,k} \\ 1 - y_{\#\{\text{cells in } V_{*,k} \text{ with no entry } j \leqslant i \text{ in, to the right or on top}\}}, & \text{otherwise.} \end{cases}$$

Here we use the convention that $y_0 = 0$.

Example 15. Figure 6 gives the example of a state in $\mathcal{T}_4^4$, and all possible states that it can transition to (with transition probabilities below the corresponding arrows). If we call V the topmost tableau, we have for example $C_V(3,1) = 1 - y_0 = 1$ and $C_V(2,2) = z_2$.

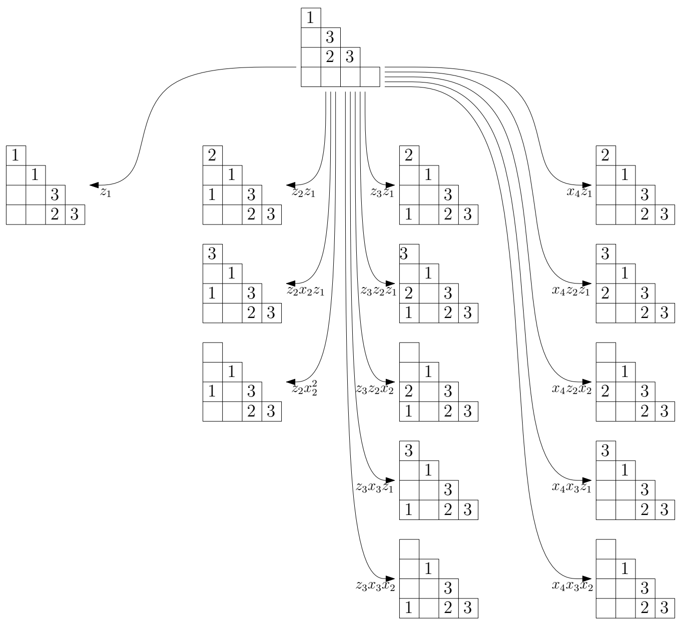

FIGURE 6. A state and its successors. Here $x_i = 1 - y_{i-1}$.

The probability of such a transition $V \to W$ is then given by

$$(46)\quad \mathcal{P}_{V,W} = \begin{cases} 0 & \text{if } W_{i,j} \neq V_{i-1,j-1} \text{ for some } 2 \leqslant j \leqslant i \leqslant n \\ \prod_{i=1}^{T-1} \mathcal{C}_W(i,1) & \text{otherwise} \end{cases}$$

This chain lumps to the overwriting model by the following procedure. Let the rows of the tableaux be numbered from bottom to top. For V be a tableau in $\mathcal{T}_n^T$, we define, for $k \in \{1, \ldots, n\}$,

$$(47)\qquad a_V(k) = \begin{cases} \text{leftmost entry on the } n-k+1\text{-th row of } V & \text{if this row is not empty} \\ T & \text{otherwise.} \end{cases}$$

The resulting lumped word $w \in St_n^T$ is then given by

$$w = a(V) := a_V(1) \cdots a_V(n). \tag{48}$$

One can check that this procedure satisfies all the conditions for lumping, see Remark 1. We are now in a position to prove Theorem 14.

Proof of Theorem 14. The probability of having a j in the last site of $w \in St_n^T$ in the overwriting chain is the same as the probability of having a j in the topmost cell of $V \in \mathcal{T}_n^T$ if $j < T$, or of having nothing in this cell if $j = T$. This means that any number $i < j$ failed to reach the first available cell, which happens with probability $1 - z_1$ for each one of them, and if $j < T$, j reached that cell, which happens with probability z_1, which proves the first part of the theorem.

For the second part of the theorem, we will only treat the case $i < j < T$, since all the cases are proved in a similar fashion. The joint probability of having a i in the $(n-1)^{th}$ site and a j in the n^{th} site of $w \in St_n^T$ is the same as the probability of being in one of the two configurations of Figure 7.

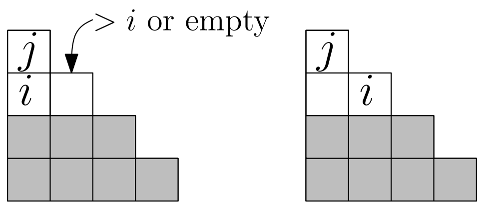

FIGURE 7. The two possible cases for $i < j < T$.

The transition probability into the first configuration is

$$\underbrace{(1-z_1)^i}_{\substack{\text{no } k \leqslant i \text{ in the topmost} \\ \text{cell of the } 2^{\text{nd}} \text{ column}}} \times \underbrace{(1-y_2)^{i-1}}_{\substack{\text{no } k < i \text{ in the top 2} \\ \text{cells of the } 1^{\text{st}} \text{ column}}} \times \underbrace{z_2}_{\substack{i \text{ is in the } 2^{\text{nd}} \\ \text{cell of the } 1^{\text{st}} \text{ column}}} \tag{49}$$
$$\times \underbrace{(1-z_1)^{j-i-1}}_{\substack{\text{no } k \text{ between } i+1 \text{ and } j-1 \text{ in the} \\ \text{topmost cell of the } 1^{\text{st}} \text{ column}}} \times \underbrace{z_1}_{\substack{j \text{ is in the topmost} \\ \text{cell of the } 1^{\text{st}} \text{ column}}} = (1-z_1)^{j-1}(1-y_2)^{i-1} z_1 z_2,$$

while the transition probability into the second configuration is

$$\underbrace{(1-z_1)^{i-1}}_{\substack{\text{no } k < i \text{ in the topmost} \\ \text{cell of the } 2^{\text{nd}} \text{ column}}} \times \underbrace{z_1}_{\substack{i \text{ is in the topmost} \\ \text{cell of the } 2^{\text{nd}} \text{ column}}} \times \underbrace{(1-y_2)^{i-1}}_{\substack{\text{no } k < i \text{ in the top 2} \\ \text{cells of the } 1^{\text{st}} \text{ column}}} \tag{50}$$
$$\times \underbrace{(1-z_1)^{j-i}}_{\substack{\text{no } k \text{ between } i \text{ and } j-1 \text{ in the} \\ \text{topmost cell of the } 1^{\text{st}} \text{ column}}} \times \underbrace{z_1}_{\substack{j \text{ is in the topmost} \\ \text{cell of the } 1^{\text{st}} \text{ column}}} = (1-z_1)^{j-1}(1-y_2)^{i-1} z_1^2,$$

By summing (49) and (50), we get the desired probability. □

The idea of this proof also hints at why the stationary distribution of the overwriting Markov chain is not of a simple form. However, we will show that the stationary distribution of the enriched Markov chain on staircase tableaux has a particularly nice structure.

Theorem 16. *The stationary distribution of $V \in \mathcal{T}_n^T$ for the staircase tableaux enriched Markov chain is given by*

$$\Pi(V) = \prod_{k=1}^{n} \prod_{i=1}^{T-1} \mathcal{C}_V(i,k)$$

with the normalisation factor $(z_1 + \cdots + z_{n+1})^{n(T-1)} = 1$.

We will prove this formula by considering an even larger enlargement of the Markov chain on staircase tableaux, analogous to the doubly enriched chain of the single species annihilation model in [1, Definition 4.14]. An immediate corollary of this result is the formula for the stationary distribution of the overwriting Markov chain

Corollary 17. *The stationary probability of $w = w_1 \cdots w_n \in St_n^T$ for the overwriting model is given by*

$$\pi(w) = \sum_{\substack{V \in \mathcal{T}_n^T \\ a(V)=w}} \Pi(V). \tag{51}$$

where $a(V)$ is defined in (48).

3.3.3. *The doubly enriched chain.* In this section we will construct a generalization of the doubly enriched chain of the single species annihilation model in [1, Definition 4.14], which was a Markov chain on words of length n. In this case, the natural extension of this Markov chain is to matrices. The state space $\tilde{\mathcal{T}}_n^T$ is the set of matrices with $T-1$ rows and n columns with entries in $\{1, \dots, n+1\}$.

It is clear there are $(n+1)^{n(T-1)}$ different states. As usual, given $z_1, \dots, z_{n+1}$ nonnegative real numbers summing to 1, the transitions are defined as follows. For $M \in \tilde{\mathcal{T}}_n^T$, all transitions from M are obtained by deleting the last column in M, shifting all the remaining columns to the right by one, and adding an arbitrary column on the left. The probability of this transition is given by the product of factors z_i for each element i in the resulting first column. More precisely,

$$\tilde{\mathcal{P}}_{M,N} = \begin{cases} \prod_{i=1}^{T-1} z_{N_{i,1}} & \text{if } N_{k,l} = M_{k,l-1} \text{ for all } k \in \{1, \dots, T-1\} \\ & \qquad \text{and all } l \in \{2, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases} \tag{52}$$

Since this is a product of $T-1$ independent copies of the single row Markov chain, it is clear this Markov chain is recurrent. Further, the stationary distribution is given, for $M \in \tilde{\mathcal{T}}_n^T$, by the product

$$\tilde{\Pi}(M) = \prod_{i=1}^{T-1} \prod_{j=1}^{n} z_{M_{i,j}}. \tag{53}$$

with normalisation $(z_1 + \cdots + z_n)^{n(T-1)} = 1$.

Remark 18. The dynamics of the doubly enriched chain guarantee that the stationary distribution is reached after n steps (indeed, the first state has

been completely forgotten after n steps), which is the desired ultrafast convergence property. Equivalently, n is a deterministic strong stationary time for this chain.

An immediate consequence of Remark 18 is a complete description of the spectrum of the transition matrix, given by the following theorem.

Theorem 19. *Let $n, T \in \mathbb{N}$, and let M be the transition matrix of the doubly enriched chain on $\tilde{\mathcal{T}}_n^T$. The eigenvalues for M are 1 with multiplicity 1 and 0 with multiplicity $n-1$.*

Proof. As stated before, the stationary distribution is reached after n steps. This means that M^n is the Matrix with all the rows being the left normalised eigenvector for M (which represents the stationary distribution). Thus, $M^{n+1} = M^n$, and therefore $X^{n+1} - X^n$ is a nullifying polynomial for M. This shows that 1 is an eigenvalue of multiplicity 1 (we already knew that its multiplicity was at least 1) and that 0 is the only other eigenvalue. $\square$

The doubly enriched Markov chain on matrices described above lumps onto the singly-enriched chain on staircase tableaux. Let M be in $\tilde{\mathcal{T}}_n^T$. We will construct a tableau T associated to M by starting with an empty tableau. We then fill T according to the following pseudocode.

- for k decreasing from n down to 1:
  - for i increasing from 1 to $T-1$:
    - if $M_{i,k}$ is less than or equal to the number of available positions in the k'th column of T:
      - insert i in the $M_{i,k}$'th position from the top
    - else:
      - do not insert i.

We then set $A(M) := T$. One can check that this algorithm leads to an actual lumping between these two Markov chains, see Remark 1.

Example 20. Figure 8 gives the example of a matrix M in $\tilde{\mathcal{T}}_4^3$ and the tableau T in $\mathcal{T}_4^3$ it lumps onto. This tableau T can then be lumped onto $w = 1243$ in St_4^3.

$$\begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 5 & 5 & 2 \\ 1 & 4 & 1 & 1 \end{pmatrix} \qquad \begin{array}{|c|cccc} \hline 3 & \multicolumn{1}{|c}{} & & & \\ \hline & \multicolumn{1}{|c|}{} & & & \\ \hline 2 & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{3} & & \\ \hline & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{1} & \\ \hline \end{array}$$

FIGURE 8. A matrix and the tableau it lumps to.

Remark 21. Since lumpings preserve the ultrafast convergence property, both the chain on staircase tableaux on $\mathcal{T}_n^T$ and the overwriting Markov chain on St_n^T converge in n steps.

We now prove the formula for the stationary distribution of the Markov chain on staircase tableaux.

Proof of Theorem 16. We have to check that, for each $V \in \mathcal{T}_n^T$,

$$\sum_{\substack{M \in \tilde{\mathcal{T}}_n^T \\ M \in A^{-1}(V)}} \tilde{\Pi}(M) = \Pi(V)$$

To do so, let $k \in \{1, \ldots, n\}$ and $i \in \{1, \ldots, T-1\}$. If i appears in the k^{th} column of V, then every M in $\tilde{\mathcal{T}}_n^T$ projecting to V, must verify

$$M_{i,k} = \#\{\text{cells above entry } i \text{ in } V_{*,k}, \text{ with no entry } j \leqslant i \text{ to the right}\}$$

because of how the projection is defined. Similarly if i does not appear in the k^{th} column of V, then $M_{i,k}$ must be greater than

$$\#\{\text{cells in } V_{*,k} \text{ with no entry } j \leqslant i \text{ in, to the right or atop of them}\}.$$

In both cases, summing z_ℓ over all the possible values ℓ of $M_{i,k}$ gives us $C_V(i,k)$, and thus proves the result. □

4. Several jugglers

We now consider a completely different generalization of Warrington's model [13]. Instead of a multivariate or multispecies generalization, we will now consider that there are several jugglers, and that each one of them can send the balls she catches to any other juggler. We model this situation as follows. For r, c, ℓ nonnegative integers such that $\ell \leqslant rc$, we denote by $S_{r\times c}$ the set of rectangular arrays with r rows and c columns, such that each cell either is empty or contains a ball, and by $S_{r\times c,\ell} \subset S_{r\times c}$ the subset of arrays containing exactly ℓ balls. Each column represents the balls that are sent to a specific juggler. For A and B two arrays in $S_{r\times c}$, we denote A^- the array obtained by removing all the balls in the lowest row, and moving all the other balls down one row (hence the topmost row of A^- is always empty). We write $A \subset B$ if all the balls in A are also in B. For i between 1 and r, we denote by A_i the number of balls in the i-th row (rows are numbered from top to bottom).

The several jugglers Markov chain is the Markov chain on the state space $S_{r\times c,\ell}$ whose transition probabilities read, for $A, B \in S_{r\times c,\ell}$,

$$\mathcal{P}_{A,B} = \begin{cases} \dfrac{1}{\binom{rc-\ell+A_r}{A_r}} & \text{if } A^- \subset B, \\ 0 & \text{otherwise.} \end{cases} \tag{54}$$

Here, A_r is the number of balls in the lowest row of A, which is exactly the number of balls the jugglers will have to send back. These A_r balls are reinjected uniformly in the $rc - \ell + A_r$ available positions, under the constraint that no two balls go to the same position. Note that there are no balls reinjected when $A_r = 0$, and $\mathcal{P}_{A,A^-} = 1$ in this case. The irreducibility and aperiodicity of the several jugglers Markov chain are easy to check.

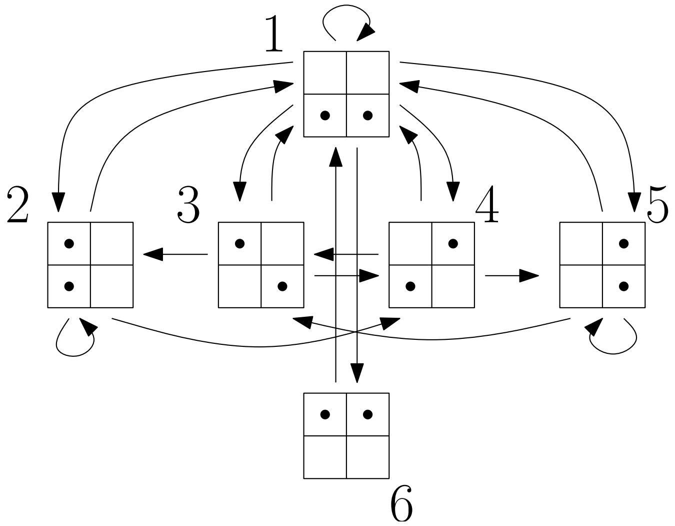

FIGURE 9. The several jugglers Markov chain on the set space $S_{2\times 2,2}$.

Example 22. The transition Matrix of the several jugglers Markov chain on the set space $S_{2\times 2,2}$ in the basis ordered $(1,2,3,4,5,6)$ on Figure 9 reads

$$\begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{55}$$

Note that $(6,3,3,3,3,1)$ is a left eigenvector for the eigenvalue 1.

Again, we have an explicit expression for the stationary distribution of this Markov chain.

Theorem 23. *The stationary probability of $A \in S_{r\times c,\ell}$ for the several jugglers Markov chain reads*

$$\pi(A) = \frac{1}{Z_{r\times c,\ell}} \prod_{i=1}^{r} (ci - A_{<i})_{A_i} \tag{56}$$

where $A_{<i} = A_1 + \cdots + A_{i-1}$ is the number of balls strictly above row i, $(x)_n = x(x-1)\cdots(x-n+1)$ is the Pochhammer symbol and $Z_{r\times c,\ell}$ is the normalization factor.

Remark 24. We didn't find a simple expression for the normalization factor $Z_{r\times c,\ell}$.

Proof. We introduce an enriched chain as follows. A state in the enriched chain is an $(r+1)\times c$ array with ℓ arcs, each arc going between two cells that are not in the same row. For each arc we mark the top cell with a cross and the bottom cell with a ball. Each cell can contain at most one cross and at most one ball, but could have one ball and one cross belonging to different arcs. The projection to $S_{r\times c,\ell}$ is obtained by simply removing the top row,

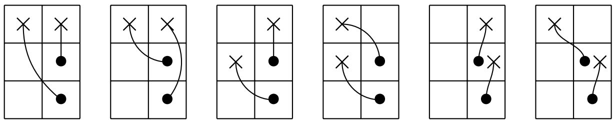

FIGURE 10. All the states in the enriched chain projecting to the state in $S_{2\times 2,2}$ with both balls in the right column.

all arcs and crosses but leaving the balls in place. See Figure 10 for all the states of the enriched chain projecting down to the rightmost state in Figure 9.

The transitions in the enriched chain are obtained by first moving all arcs, balls and crosses down one row in the array. Secondly, if there were balls in the bottom row, they and the corresponding arcs and crosses are removed. The balls are reinjected uniformly into the array except for the top row and under the condition that no two balls may be in the same cell, just like in the several jugglers Markov chain. For each of these balls an arc is inserted from the ball and up to a cross positioned uniformly in the top row under the condition that no two crosses can be in the same cell. (Alternatively we could define the transitions such that the new crosses in the top row appear in the same columns as the removed balls.)

Now we note that if we run the enriched chain backwards, it will be an identical chain with the roles of balls and crosses exchanged (turned upside down). The number of balls in the bottom row in a state is equal to the number of crosses in the top row for every state it may transition to. Also the number of ways to inject balls is the same as the number of ways of removing crosses. It follows that the number of transitions out of any state is equal to the number of transitions into the same state. Thus the stationary distribution is uniform for the enriched chain.

The uniformity of the enriched chain means that, to evaluate the stationary probability $\pi(A)$ of a state of the several jugglers Markov chain, it suffices to count the number of states in the enriched chain projecting to it. For each ball we can place the cross in any position in a row above with the constraint that no two crosses can be in the same cell. The number of possibilities can be counted row by row: assuming that the crosses corresponding to the balls strictly above row j have been chosen, there remains $cj - A_{<j}$ cells without crosses that may be matched with the A_j balls in row j, hence there are $(cj - A_{<j})_{A_j}$ possible choices for row j.

□

Remark 25. In [10], J.S. Kim studied the model of a juggler with each site being allowed to countain up to a certain number $c > 1$ of balls. This model can be obtained from the several jugglers Markov chain by a lumping.

5. Open Problems

Several questions remain open in the multispecies juggling context. We have not found an expression for the normalization factor for the juggling chain with several jugglers. We have also not yet found a multiparameter version for the latter model, as the possibility of catching more than one ball

at a time changes the behavior quite drastically. A multispecies model with several jugglers is one possible extension of our model. From a probabilistic point of view, it would also be natural to look atthe extension to infinite models, such as a Markov chain on the set space $St_{n_1,\ldots,n_{T-1},\infty}$. This would contain as special cases, the infinite and unbounded juggling models [1].

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